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The Thy-Angle and g-Angle in a Quasi-Inner Product Space

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Abstract. In this note we prove that in a so-called quasi-inner product spaces, introduced a new angle (Thy-angle) and the so-called g-angle (previously defined) have many common characteristics. Important statements about parallelograms that apply to the Euclidean angles in the Euclidean space are also valid for the angles in a q.i.p. space (see Theorem 1).

1. Introduction

Let $(X, \|\cdot\|)$ be a seminormed space. Based an idea of I. Singer [5], Volker Thürey in [6] introduced a new concept of an angle between elements x and y of $X\setminus\{0\}$, so-called Thy-angle ($\angle_{Thy}(x,y)$), as follows:

$$(1)\qquad \angle_{Thy}(x,y) := \arccos\left[\frac{1}{4}\left(\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|^2-\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2\right)\right].$$

This new angle corresponds with the Euclidean angle in the case that $(X, \|\cdot\|)$ already is an inner product space. In the real normed space $(X, \|\cdot\|)$, $\dim X > 1$, $x, y \neq 0$, for this angle we have the properties:

1) $\angle_{Thy}$ is a continuous surjective function from $[X\setminus\{0\}]^2$ to $[0,\pi]$,
2) $\angle_{Thy}(x,x) = 0$,
3) $\angle_{Thy}(-x,x) = \pi$,
4) $\angle_{Thy}(x,y) = \angle_{Thy}(y,x)$,
5) $\angle_{Thy}(rx,sy) = \angle_{Thy}(x,y)$, $r,s>0$,
6) $\angle_{Thy}(-x,-y) = \angle_{Thy}(x,y)$,
7) $\angle_{Thy}(x,y) + \angle_{Thy}(-x,y) = \pi$.

With this angle we observe here is another angle was previously defined, which we now define.

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It is well known that in a real smooth normed space $(X, \|\cdot\|)$, always exists the functional

$$(2) \qquad g\,(x,y) := \|x\| \lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \qquad (x,y \in X)\,.$$

(see [1]).

This functional is linear in second argument and it has the following properties:

$$(3) \qquad \begin{array}{ll} g\,(rx,y)) = rg\,(x,y)\,, & g\,(x,x) = \|x\|^2\,, \\ |g\,(x,y)| \leqslant \|x\|\,\|y\|\,, & (x,y\in X;\ r\in R). \end{array}$$

In an arbitrary normed space, we are in [2] define another angle, so-called g-angle with

$$(4) \qquad \angle_g\,(x,y) := \arccos \frac{g\,(x,y) + g\,(y,x)}{2\,\|x\|\,\|y\|}, \qquad (x,y\in X\backslash\,\{0\})\,,$$

and the so-called g-orthogonality vectors with

$$x\bot_g y \Leftrightarrow g(x,y) + g(y,x) = 0, \qquad (x,y\in X\backslash\,\{0\}\,.$$

Let us mention there that the so-called Pythagorean orthogonality vectors defines

$$x\bot_P y \Leftrightarrow \|x\|^2 + \|y\|^2 = \|x+y\|^2\,, \qquad (x,y\in X\backslash\,\{0\}).$$

Also note that known the Singer orthogonality can be defined with

$$x\bot_S y \Leftrightarrow \angle_{Thy}(x,y) = \frac{\pi}{2}.$$

A normed space $(X, \|\cdot\|)$ of property

$$(5) \quad \|x+y\|^4 - \|x-y\|^4 = 8[\|x\|^2\,g\,(x,y) + \|y\|^2\,g\,(y,x)], \qquad (x,y\in X)$$

we call a quasi-inner product space (q.i.p space) (see [3]).

The space of sequences l^4 is a q.i.p. space

$$\left(x = (x_k)\,, \quad y = (y_k) \in l^4, \quad g(x,y) = \|x\|^{-2} \sum_k |x_k|^3\,(\operatorname{sgn} x_k) y_k) \right),$$

but l^1 is not a q.i.p. space (see [3]).

In [4] we have proved that, in a q.i.p. space

$$x\bot_g y \Leftrightarrow x\bot_S y \quad (x,y\in X\backslash\,\{0\}).$$

Having regard to the equality (5) comparing the definition (1) and (4) we conclude that there is a direct link between g-angle and Thy-angle.

Namely, for $x, y \neq 0$, if instead x it takes place $\frac{x}{\|x\|}$ and instead y take $\frac{y}{\|y\|}$, from (5) we get

$$\begin{aligned}(6) \qquad &\frac{1}{4}\left[\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|^2+\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2\right]\cdot\\ &\cdot\frac{1}{4}\left[\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|^2-\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2\right]=\\ &=\frac{g(x,y)+g(y,x)}{2\,\|x\|\ \|y\|}.\end{aligned}$$

From this equality and (3) we conclude that

$$\begin{aligned}-1 \leq &\frac{1}{4}\left[\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|^2+\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2\right]\cdot\\ &\cdot\frac{1}{4}\left[\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|^2-\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^2\right]\leqslant 1.\end{aligned}$$

This means that in a q.i.p. spaces X can be define another angle between vectors $x, y \in X \setminus \{0\}$ with

$$(7) \qquad \angle(x,y) := \arccos\frac{1}{16}\left[\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|^4-\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|^4\right].$$

In fact in a q.i.p. space, this angle is equal to g-angle.

Knowing that

$$g(rx, sy) = g(x,y) \quad (r, s \neq 0)$$

it can be seen instead $\angle(x,y)$ the angle $\angle(u,v)$, where

$$u = \frac{x}{\|x\|},\ v = \frac{y}{\|y\|} \in S(X)$$

($S(X)$ is the unit sphere of X).

Then (6) becomes

$$(8) \quad \frac{1}{4}[\|u+v\|^2+\|u-v\|^2]\cdot\frac{1}{4}[\|u+v\|^2-\|u-v\|^2] = \frac{g(u,v)+g(v,u)}{2}.$$

Checking more

$$k = \frac{1}{4}[\|u+v\|^2+\|u-v\|^2],$$
$$b = \frac{1}{4}[\|u+v\|^2-\|u-v\|^2],$$
$$a = \frac{g(u,v)+g(v,u)}{2}$$

get

$$(9) \qquad kb = a,$$

i.e., for all $u, v \in S(X)$, $\arccos kb = \arccos a$.

This means that, for $x, y \in X \setminus \{0\}$

$$\angle(x, y) = \angle_g(x, y).$$

Although $\angle_g(u, v)$ and $\angle_{Thy}(u, v)$ are two mutually different functional they, in a q.i.p. space, have many common characteristics.

Modeled in terms of Euclidean geometry, we adopt the following terminology in normed spaces.

From now on we assume that points 0, x, y are the vertices of the triangle $(0, x, y)$ and points 0, x, y, $x + y$ are the vertices of the parallelogram $(0, x, y, x+y)$. The numbers $\|x + y\|$, $\|x - y\|$ are the lengths of diagonal of this parallelogram. If $\|x\| = \|y\|$, we say that this parallelogram is a romb, and if $x \perp_\rho y$ we say that the parallelogram $(0, x, y, x + y)$ is a $\rho-$ rectangle.

2. MAIN RESULTS

Justification for introducing these angles in the normed spaces show, among other things, the following Theorem 1.

Theorem 1. *Let X be a q.i.p. space. The following statements are true:*

a) *The g-angle has properties 1)-7), similar to the Thy-angle;*
b) *The lengths of diagonals parallelogram $(0, x, y, x+y)$ are equal if and only if this parallelogram is Thy-rectangle, i.e. $x \perp_S y$ or $\angle_{Thy}(x, y) = \pi/2$;*
c) *The diagonals of the romb $(0, x, y, x + y)$ are Thy-orthogonal, i.e. $(x - y) \perp_S (x + y)$;*
d) *The parallelogram $(0, x, y, x + y)$ is a Thy-quadrangle if and only if its lengths of the diagonals are equal and the diagonals are Thy-orthogonal.*

Proof. Using the properties (3) of g-functional easy to check these properties 1)-7) are valid for the g-angle.

For evidence statements b)-d) to use gender The-orthogonality of the g-orthogonality ($\perp_{Thy} = \perp_g$) since the g-orthogonality assertion is proved in [4]. □

Following two theorems show the relationship of these two angles depending on the vectors $u, v \in S(X)$.

Theorem 2. *Let X be a q.i.p. space. The following statements are true:*

1. *For all $u, v \in S(X)$ it is $\operatorname{sgn} a = \operatorname{sgn} b$, i.e.,* $\operatorname{sgn} \angle_g(u, v) = \operatorname{sgn} \angle_{Thy}(u, v)$,
2. $\angle_g(u, v) = \angle_{Thy}(u, v) \Leftrightarrow (u + v) \perp_P (u - v) \vee u \perp_g v$.

Proof.
1. Since the $k > 0$ from (9) we get $\operatorname{sgn} a = \operatorname{sgn} b$.

2. According to the definitions (1) and (4) we have (9) so

$$\angle_g(u,v) = \angle_{Thy}(u,v) \Leftrightarrow k = 1 \vee \|u+v\| = \|u-v\|\,.$$

If $k = 1$ then

$a = b \wedge \|u+v\|^2 + \|u-v\|^2 = 4 = \|(u+v)+(u-v)\|^2 \Leftrightarrow (u+v \perp_P (u-v)$.

Since $k > 0$ it is

$$\|u+v\| = \|u-v\| \Leftrightarrow a = b = 0 \Leftrightarrow \angle_g(u,v) = \angle_{Thy}(u,v).$$

The interrelation of angles $\angle_g(u,v)$ and $\angle_{Thy}(u,v)$ depends on the relationship between length of diagonals of a parallelogram $(0, u, v, u+v)$. □

Theorem 3. *Let X be a q.i.p. space. The following assertion are valid:*

1. *If $\|u-v\| < \|u+v\|$ then*

$$\|u+v\|^2 + \|u-v\|^2 > 4 \quad \Rightarrow \quad \angle_g(u,v) \ < \ \angle_{Thy}(u,v),$$
$$\|u+v\|^2 + \|u-v\|^2 < 4 \quad \Rightarrow \quad \angle_g(u,v) \ > \ \angle_{Thy}(u,v).$$

2. *If $\|u-v\| > \|u+v\|$ then*

$$\|u+v\|^2 + \|u-v\|^2 > 4 \quad \Rightarrow \quad \angle_g(u,v) \ > \ \angle_{Thy}(u,v),$$
$$\|u+v\|^2 + \|u-v\|^2 < 4 \quad \Rightarrow \quad \angle_g(u,v) \ < \ \angle_{Thy}(u,v).$$

Proof.
1. Since the $k > 1$ and $b > 0$ according to (9) we have

$$a = kb > b, \text{ so } \arccos a < \arccos b, \text{ i.e., } \angle_g(u,v) < \angle_{Thy}(u,v).$$

If $k < 1$ and $b > 0$ then $a = kb < b$ so $\arccos a > \arccos b$,
i.e., $\angle_g(u,v) > \angle_{Thy}(u,v)$.

2. Since $k > 1$ and $b < 0$ get $a = kb < b \quad \Rightarrow \quad a < b$, so

$$\arccos a > \arccos b \quad \Leftrightarrow \quad \angle_g(u,v) \quad > \quad \angle_{Thy}(u,v).$$

$$k < 1 \wedge b < 0 \quad \Rightarrow \quad a = kb > b \quad \Rightarrow \quad a > b \quad \Leftrightarrow$$
$$\arccos a < \arccos b \quad \Leftrightarrow \quad \angle_g(u,v) \quad < \quad \angle_{Thy}(u,v).$$ □

References

[1] P.M. Miličić, *Sur le produit scalaire géneralisé*, Mat. Vesnik, **(25)10)** (1973), 325-329.

[2] P.M. Miličić, *Sur la g-angle dans un espace normé*, Mat. Vesnik, **45** (1993), 69-77.

[3] P.M. Miličić, *A generalization of the parallelogram equality in normed spaces*, Jour. Math. Kyoto Univ (JMKYAZ), **38(1)** (1998), 71-75.

[4] P.M. Miličić, On the quasi-inne product spaces, Matematički Bilten, **22(XLVIII)** (1998), 19-30.

[5] I. Singer, *Unghiuri Abstracte si Functii Trigonometrice in Spatii Banach*, Buliten Stintefic, Sectia de Stiinte Matematice si Fisice, Academia Republicii Populare Romine, **9** (1957), 29-42.

[6] V. Thürey, *Angles and Polar Cordinates in Real Normed Spaces*, arXiv:0902.2731v2.

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